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ON THE SET COINCIDENCE GAME*

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In the Set Coincidence Game $G(V, W)$, two players alternately choose elements not previously chosen from a finite, nonempty set V, and W is a given family of nonempty subsets of V (the 'winning sets'). The winner is that player who first adds an element to the set of 'chosen' elements S, so that $S \in W$. This game is closely related to and generalizes Ringeisen's Isolation Game on graphs. We develop the theory of $G(V, W)$, present and support a conjecture about the structure of minimal forced wins, and then prove a weakened form (the Weak Filter Theorem). It is hoped that the indicated themes about optimal design of forced wins will prove of interest for a variety of combinatorial games.

1. Introduction

The *Set Coincidence Game* $G(V, W)$, a generalization of the Isolation Game of Ringeisen [9], is played on a finite nonempty set V of *elements*. W is a collection of nonempty subsets of V, the *winning* sets. Players P1 and P2 move alternately, with P1 leading off; at each turn, a player adds a new element to an expanding set S, which was empty at the start of play. If a player's move causes S to coincide with some $w \in W$, then that player *wins* (the opponent loses), and play ends. If V is exhausted (i.e., $S = V$) without a win, then the game is drawn. The fact that both players' choices contribute to building up a *single* set S, rather than individual sets S^1 and S^2, suffices to differentiate $G(V, W)$ from the more-studied 'positional games of types 1 and 2' as defined by Berge [4], and called 'amoeba games' (weak and strong) in Beck and Csirmaz [2], which in turn include most of the 'achievement and avoidance' games of Harary (e.g. [7, 8]). On the other hand, the diameter and geodesic achievement games of Buckley and Harary [5, 6] *are* set coincidence games.

$G(V, W)$ will be called a *forced p-win* if one of the players has a strategy assuring a win in no more than p moves, but the opponent has at least one way to prolong play to a full p moves. (Thus $p \leqslant n = |V|$, and the winner is P1 or P2 according as p is odd or even.) If $G(V, W)$ is a forced p-win for some p, we call it a *forced win*.

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The Isolation Game of Ringeisen [9], which this game generalizes, is played beginning with a graph $H = (V, E)$; the players alternately modify the graph, with the objective of being the first to isolate a vertex. The 'modifications' allowed by the rules will not be repeated here, but do not explicitly define $I(H)$ as a Set Coincidence Game; we have shown in [10], however, that $I(H) = G(V, W)$ where W consists of all vertex-neighborhoods in H plus their complements. This observation proved sufficient in [10] to permit the analysis of $I(H)$ for a number of classes of graphs H, but a fully general analysis was thwarted by our inability to find a proof-facilitating recursive structure: the result of a partial play of $I(H)$ does not seem to correspond to any $I(H')$, a consequence of the 'symmetry-spoiling' presence of each $v \in V$ in the complement of its neighborhood. This motivated imbedding the Isolation Games in a larger class of games which *do* admit recursive treatment.

That the games $G(V, W)$ indeed form such a class is readily seen. For, consider a partial play of $G(V, W)$ which has not yielded a win, and as above, let S denote the set of elements selected so far (by both players). Then the resultant *continuation game*, denoted $G(V, W, S)$, is readily seen to coincide with the game $G(V - S, W_S)$ where $W_S = \{w - S : w \in W, S \subset w\}$. (This is precisely the notion, Berge [3], of 'induced hypergraph'.)

In particular, $G(V, W)$ is a forced win for P1 iff either (i) it is a forced 1-win (i.e., W includes a singleton) or (ii) at least one of the continuation games $\{G(V, W, \{v\}) : v \in V\}$ is a forced win for its *second* player. Since checking W for singletons can be regarded as trivial, we see that the problem of determining whether forced winnability by P1 holds, can be reduced to the corresponding problem (on a smaller game) for P2. Thus what follows will often concentrate on the latter problem, for example explicitly assuming p *even* in most of our discussions of forced p-wins, with assurance that no loss of generality can result.

Except where continuation games are involved, the set V of elements affects $G(V, W)$ only via its cardinality n, and the above mentioned recursive arguments will involve induction on n. We will therefore generally write $G(n, W)$ instead of $G(V, W)$, implicitly assuming $V = \{1, 2, \ldots, n\}$, when no ambiguity is possible.

A natural first question is: given any particular $G(n, W)$, is it a forced win for P1 or a forced win for P2 or a draw? In [11] we showed—for a 'tightened' encoding of $G(n, W)$—that the decision problem for forced-winnability by P2 is PSPACE-complete.

It has proved fruitful to consider also the combinatorial optimization problem of main concern here:

$$\Pi(n, p): \quad \min\{|W| : G(n, W) \text{ a forced } p\text{-win}\},$$

which might confront a game *designer* required to produce a forced p-win using a limited allowance of winning sets. The case $p = n$ is trivial (just take $W = \{V\}$), so we assume $p < n$ throughout. Analysis of $\Pi(n, p)$ is aided by visualizing the Hasse diagram of subsets of V as a digraph D_n, with an arc from each node at *level* λ of the digraph (these nodes are just the λ-sets of V) for $\lambda = 0, 1, \ldots, n-1$, to each of the $(\lambda+1)$-sets that contains it. Each play of $G(n, W)$ corresponds to a *path* in D_n, beginning at the root-node ($\emptyset$) of D_n and rising through nodes S_λ at successive levels λ until terminated either by reaching some $w \in W$ (a win) or by winlessly reaching the single level-n node V (a draw). Thus S_λ denotes the 'value' of the expanding set S just after move λ. We use the term 'trajectory' to denote the sequence of subsets of *even* cardinality encountered along a path. Note that feasibility of $\Pi(n, p)$ is not in question, since choosing W to consist of all level-p nodes certainly yields a forced p-win.

In [11], we showed that in game $G(n, \emptyset)$, the minimum width for any fixed strategy of P2 of the tree of 'attainable' play-trajectories (corresponding to the different strategies for P1) increases rapidly as the tree rises from level to level, until the mid-level $\lceil n/2 \rceil$ is reached. This suggests the intuition that unless $n-p$ is small (p even), an optimal solution W of $\Pi(n, p)$ must place its meager number of winning sets within D_n so as to *limit play* at the lower levels to just a very few trajectories, in the sense that deviations by the winning player P2 are punished by 'losing the win' (permitting the opponent to draw or win), while deviations by P1 are punished by premature loss. Accordingly, in Section 2, we characterize those feasible solutions W of $\Pi(n, p)$—to be called *p-filters*—which (roughly speaking) minimize $|W|$ subject to the *further* restriction of limiting play at the first $p-2$ levels to just a *single* trajectory. The preceding 'intuition' is then formalized by a precise statement of the *Filter Conjecture*: *unless n-p is small, the optimal solutions of $\Pi(n, p)$ are precisely the p-filters.*

We have not succeeded in proving the Filter Conjecture, and offer its general case as a challenging open problem. Section 3 contains our (increasingly complicated) verifications of its low-order cases $p = 2, 4, 6$. Fortunately, these cases are adequate to provide most of the induction base for establishing, in Section 4, the following weaker result: for even $p \geq 8$, unless $n-p$ is small, $n+3$ is a lower bound for the optimal value of $\Pi(n, p)$. As will be shown in a subsequent paper (based on Chapter 5 of [13]), this more limited result is sufficient to permit completing our analysis of the isolation games $I(H)$, with the surprising outcome that (apart from a few identified possible exceptions) these games can be forced-won only either very *early* ($p \leq 5$) or very *late* ($p = n-2$).

Before beginning the body of the paper, we remind the reader of the notation S_λ defined above, and introduce the notation W_λ for the family of winning λ-sets in $G(n, W)$. The complement of a set B, with respect to some context-specified superset, will be denoted B^c.

2. Covers, forced wins, filters

We turn to general properties of $G(n, W)$ that are relevant to the problem $\Pi(n, p)$ posed in the introduction. The material will lead up to a statement of the

Filter Conjecture, which formalizes the 'intuition' expressed near the end of Section 1.

For any set Q of elements, a winning set $w = Q \cup \{u, v\} \in W_{|Q|+2}$ is said to *cover* each of the adjunctions u, v to Q; note that if play reaches $S = Q$ and a player then chooses (i.e., adds to S) either of $\{u, v\}$ without winning, the other player can win immediately via w by choosing the other member of the pair. A subcollection C of $W_{|Q|+2}$ is said to be a *cover* of Q if $V \subset \cup\{w \in C : Q \subset w\}$; if one exists, Q is said to be *covered*. That occurs if and only if $W_{|Q|+2}$ itself is a cover of Q. The next lemma illustrates the relevance of this 'coverage' concept, which is reminiscent of those of *shade* and *shadow* (see Anderson [1]) although the latter refer to a pair of levels differing by 1 rather than 2.

Lemma 2.1. *If $G(n, W)$ is a forced p-win with $p > 1$, then at least one non-winning set Q of size $p - 2$ must be covered.*

Proof. Take Q to be the set S_{p-2} arising after move $p - 2$ in a play exhibiting a forced p-move win. If Q is not covered, then $\exists v \notin \cup\{w \in W_p : S_{p-2} \subset w\}$. If the collection appearing in this union is empty (i.e., no $w \in W_p$ contains S_{p-2}) then continuation from S_{p-2} to a p-move win is impossible; if the collection were nonempty, then $v \notin S_{p-2}$ and so a choice of v at move $p - 1$ is possible, ruling out a win at move p. Either case yields a contradiction, so the lemma is proved. □

Lemma 2.2. *A cover C for a set Q of cardinality λ is of size at least $\lceil (n - \lambda)/2 \rceil$, with equality if C is properly chosen.*

Proof. The result follows immediately from the observation that each $w \in C$ can cover at most two adjunctions to Q since $|w - Q| = 2$. □

We now give a theorem which will be frequently appealed to.

Theorem 2.1. *In $G(n, W)$, if $W_{\lambda+2}$ contains covers C_1 and C_2 of two distinct sets Q_1 and Q_2 (respectively) on level λ, then $|C_1 \cup C_2| \geq n - \lambda - 1$ and so each element in $Q_1 \cap Q_2$ lies in at least $n - \lambda - 1$ members of $W_{\lambda+2}$ (at least $n - \lambda + 1$ if $|Q_1 \cup Q_2| > \lambda + 2$ with $n - \lambda$ odd; at least $n - \lambda$ if either $|Q_1 \cup Q_2| > \lambda + 2$ with $n - \lambda$ even, or $|Q_1 \cup Q_2| = \lambda + 2$ with $n - \lambda$ odd, or $|Q_1 \cup Q_2| = \lambda + 1$ with $n - \lambda - 1 - |C_1 \cap C_2|$ either negative or odd).*

Proof. Assume without loss of generality that each set in C_1, C_2 contains Q_1, Q_2 respectively, so that each set in $C_1 \cap C_2$ contains $Q_1 \cup Q_2$. $|C_1 \cup C_2| = |C_1| + |C_2| - |C_1 \cap C_2|$; by Lemma 2.2, each $|C_i| \geq \lceil (n - \lambda)/2 \rceil$.

If C_1 and C_2 are disjoint, it follows that $|C_1 \cup C_2| \geq 2\lceil (n - \lambda)/2 \rceil \geq n - \lambda$, with strict inequality for $n - \lambda$ odd. This must be the case if $|Q_1 \cup Q_2| > \lambda + 2$, since then no $w \in W_{\lambda+2}$ could satisfy $Q_i \subset w$ for $i = 1, 2$. If $|Q_1 \cup Q_2| = \lambda + 2$ then the only possible member of $C_1 \cap C_2$ is $Q_1 \cup Q_2$, so $|C_1 \cup C_2| \geq 2\lceil (n - \lambda)/2 \rceil - 1 \geq n - \lambda - 1$ with the last inequality strict if $n - \lambda$ is odd.

We may now assume $|Q_1 \cup Q_2| \leqslant \lambda + 1$; since Q_1, Q_2 are distinct λ-sets, it follows that $|Q_1 \cup Q_2| = \lambda + 1$. Let $C_1 \cap C_2 = \{w_j : j = 1, \ldots, d\}$; we may also assume $d > 0$, and can write $w_j = Q_1 \cup Q_2 \cup \{u_j\}$. Let $U = \{u_j : j = 1, \ldots, d\}$ so that $|U| = d$. If $d \geqslant n - \lambda$ then since $|C_1 \cup C_2| \geqslant |C_1 \cap C_2| = d \geqslant n - \lambda$, the Theorem holds. Otherwise, the sets in $C_1 - C_2$ must cover adjunctions to Q_1 from the $n - (\lambda + 1 + d)$ elements outside $Q_1 \cup Q_2 \cup U$, and so must number at least $\lceil (n - \lambda - 1 - d)/2 \rceil$; similarly for $C_2 - C_1$. Thus

$$\begin{aligned} |C_1 \cup C_2| &= |C_1 - C_2| + |C_2 - C_1| + |C_1 \cap C_2| \\ &\geqslant 2\lceil (n - \lambda - 1 - d)/2 \rceil + d \geqslant n - \lambda - 1 \end{aligned}$$

with strict inequality for $n - \lambda - 1 - d$ odd, and the proof is complete. □

We next begin to build results about $G(n, W)$ which are related to the subsequent Filter Conjecture. With reference to a forced p-win, we denote the parity of p by π and the winning player by P, with the other parity and other player (the loser) denoted π' and P' respectively.

Theorem 2.2. *If $G(n, W)$ is a forced p-win, $2 \leqslant p \leqslant n - 1$, then every element must appear in at least $\lfloor p/2 \rfloor$ winning sets of parity π, at levels $\leqslant p$, with at least one at level p.*

Proof. Suppose, to the contrary, that some element v appears in fewer than $\lfloor p/2 \rfloor$ winning sets of parity π, at levels $\leqslant p$. Since player P' has at least $\lfloor p/2 \rfloor$ moves before level p, it must be that for one of the levels of parity π', before level p, there is no winning set containing v on the *next* level.

Let level $t + 1$ be the first level of parity π which contains no winning set w with $v \in w$. Then $t + 1 \leqslant p$, and by the definition of t there is at least one winning set containing v on every level of parity π up to and including level $t - 1$. This requires at least $\lfloor (t-1)/2 \rfloor$ winning sets containing v on levels of parity π before level t, leaving at most (we use the fact that p, t are of opposite parity)

$$(\lfloor p/2 \rfloor - 1) - \lfloor (t-1)/2 \rfloor = \lceil (p - t)/2 \rceil - 1$$

winning sets containing v at levels of parity π in the interval $[t + 3, p]$. Player P' can play so as to prolong the game as far as move t, and by choosing v at that move if $v \notin S_{t-1}$, he can ensure that $v \in S_t$. Suppose this is done; note that P cannot win on level $t + 1$, so that $p \geqslant t + 3$. We assume that P continues to move so as to maintain the forced p-win.

Claim 1. *Player P' cannot be forced to lose on level $t + 3$, i.e., $p \geqslant t + 5$.*

Proof of Claim 1. If P' has a forced loss on level $t + 3$, then P can choose move $t + 1$ to yield a covered S_{t+1}. By Lemma 2.2, S_{t+1} (which contains v) must be a subset of at least $\lceil (n - (t + 1))/2 \rceil$ winning sets on level $t + 3$. But we have just shown there are at most $\lceil (p-t)/2 \rceil - 1$ winning sets of parity π completable from S_{t+1} at levels in $[t+3, p]$. Since $p \leqslant n-1$,

$$\lceil (p-t)/2 \rceil - 1 \leqslant \lceil (n-1-t)/2 \rceil - 1 < \lceil (n-(t+1))/2 \rceil$$

so S_{t+1} cannot be covered. Thus P cannot force a win on level $t+3$ from S_{t+1}. □ (Claim 1)

Claim 2. *P' can, by his next move $t+2$, decrease the number of completable winning sets of parity π at levels $\leqslant p$.*

Proof of Claim 2. By claim 1, P' can choose move $t+2$ to avoid all the winning sets on level $t+3$ completable from S_{t+1}. If such sets exist, this choice reduces the ranks of the completable winning sets by those avoided on level $t+3$.

If there are no winning sets on level $t+3$ completable from S_{t+1}, then P' may choose any element on move $t+2$ without losing on level $t+3$. In particular P' may choose in the complement of a winning set, completable from S_{t+1}, of the type in the claim; such a set must exist for a p-win. □ (Claim 2)

Consequence of the two claims. Play arrives at level $t+3$ with at most

$$\lceil (p-t)/2 \rceil - 2 \leqslant \lceil (n-(t+5))/2 \rceil$$

completable winning sets of parity π at levels $\leqslant p$, a number which is *strictly less* than the number $\lceil (n-(t+3))/2 \rceil$, required (Lemma 2.2) to cover S_{t+3}. So P' does not face a cover at move $t+4$, and (as in the proof of Claim 2) may choose once more so as to 'kill' at least one more completable winning set. But this leaves too few to guarantee a win at level $t+7$ and so on. This contradicts the fact that $G(n, W)$ is a forced p-win, and proves the main part of the theorem.

Finally, if some $v \in V - \bigcup W_p$, consider a play corresponding to a p-win. If $v \in S_{p-2}$ then S_{p-2} cannot be completed to a winning S_p, while if $v \in V - S_{p-2}$ then P' could choose v at move $p-1$ to avoid loss at move p, a contradiction in either case. So no such v exists. □ (Theorem 2.2)

A solution W of problem $\Pi(n,p)$, because it minimizes $|W|$, is necessarily set-inclusion minimal among that problem's feasible solutions. It is therefore apropos to observe:

Lemma 2.3. *If $G = G(n, W)$ is a forced p-win with W minimal, then W can have no winning sets on levels of parity π' or on levels $>p$. Also, no member of any W_λ can be covered by $W_{\lambda+2}$.*

Proof. Suppose G is a forced p-win with such a minimal collection W of winning sets. If W has winning sets on levels $>p$, then since player P can force a win by the pth move, the winning sets on levels $>p$ play no role in a p-move exchange. So $G(n, W - \{\text{all winning sets on levels } >p\})$ remains a forced p-win, contradicting the minimality of W.

If W has winning sets on levels of parity π', then player P can force a win while avoiding a set coincidence with a winning set whose cardinality is of parity π'. This implies that $G(n, W - \{\text{all winning sets on levels of parity } \pi'\})$ remains a forced p-win, again contradicting the minimality of W. The proof of the last assertion is similar. $\square$

Motivated by the 'intuition' expressed in Section 1, we now introduce a definition.

Definition. For $G = G(n, W)$, and even $p \leq n$, we call W *p-directive* if there exists a chain of subsets

$$\emptyset = Q_0 \subset Q_2 \subset \cdots \subset Q_{p-2} \qquad (*)$$

at the even levels $<p$, such that

(a) each $G(V, W, Q_{2i})$ is a forced $(p-2i)$-win, and

(b) for each $2i < p-2$ and each $v \in Q^c_{2i+2}$, $G(V, W, Q_{2i} \cup \{v\})$ is a forced 1-win.

To interpret this, note that condition (a) with $i = 0$ makes G a forced p-win. Play must begin with $S_0 = \emptyset = Q_0$. Suppose it has reached $S_{2i} = Q_{2i}$ where $2i < p-2$. If P1 makes any choice outside Q_{2i+2}, then by (b) she can be punished with premature loss, on move $2i+2 < p$. If she chooses one of the two members of $Q_{2i+2} - Q_{2i}$ at move $2i+1$, then by (a) for $i+1$, P2 'preserves the p-move win' by choosing the *other* member of $Q_{2i+2} - Q_{2i}$, yielding $S_{2i+2} = Q_{2i+2}$. Thus $(*)$, as a trajectory through the first $p-2$ levels, is consistent with 'best play' on both sides, and (for best play) is obligatory for P1 if adopted by P2: W 'directs' play to proceed via this trajectory.

If W minimizes $|W|$ among all p-directive families of winning sets, we call it a *p-filter*. In order to state a more concrete characterization of p-filters, we set

$$k = \lceil n/2 \rceil, \qquad p = 2m.$$

The last sentence of the following theorem implies that for p-filters the unique trajectory $(*)$ is obligatory for (not merely consistent with) best play by P2 and thus by both sides; hence the term 'p-filter'.

Theorem 2.3. *The winning-set family W for $G = G(n, W)$ is a p-filter if (for $p < 2\lfloor n/2 \rfloor$) and only if (for $p < n$)*

(i) *W gives a forced $2m$-win,*

(ii) $W = \cup \{W_{2i} : i = 1, 2, \ldots, m\}$,

(iii) $|W_{2m}| = k - m + 1$,

(iv) $|W_{2i}| = k - i$ *for* $i = 1, 2, \ldots, m-1$.

In this case, if $S_{2i} \neq Q_{2i}$ for any i in some play of G, then either $S_{2i} \in W$ or else P1 *can draw the continuation $G(V, W, S_{2i})$.*

Proof.

Necessity. First assume only that W is p-directive and that $p<n$. Then (i) holds. It follows from Lemmas 2.1 and 2.2 (the latter with $\lambda=p-2$) that $|W_p| \geqslant k-m+1$, consistent with (iii). For $0 \leqslant i \leqslant m-1$, and each of the $n-(2i+2)$ members v of Q^c_{2i+2}, it follows by defining condition (b) that $Q_{2i} \cup \{v\}$ must lie in some member of W_{2i+2}, so that $|W_{2i+2}| \geqslant \lceil (n-2i-2)/2 \rceil = k-(i+1)$, consistent with (iv).

For W a p-filter, the proof of Lemma 2.3 carries over to establish (ii). To show that minimization of $|W|$ forces equality in (iii) and (iv), it suffices to exhibit an equality-achieving p-directive family W. We take $W_\lambda=\emptyset$ for λ odd or $\lambda>p$.

If $n>p=2m$ is even, we first define $Q_{2i}=\{1, 2, \ldots, 2i-1, 2i\}$ for $0 \leqslant i \leqslant m-1$, and then satisfy condition (b) by taking $W_{2i+2}=\{Q_{2i} \cup \{2j-1, 2j\}: j=i+2, \ldots, k\}$ for $i<m-1$, with W_p a minimal cover of Q_{p-2}. Satisfaction of (iii) and (iv) is apparent, while satisfaction of condition (a) by this family $W(p, n)$ can be verified by induction, using the observation that $G(V, W(p, n), Q_{2i})$ is an instance of $G(n-2i, W(p-2i, n-2i))$.

If $n>2m$ is odd, we define the Q_{2i}'s as above, but now take $W_{2i+2}=\{Q_{2i} \cup \{2j-1, 2j\}: j=i+2, \ldots, k-1\} \cup \{Q_{2i} \cup \{2k-2, 2k-1\}\}$ for $i<m-1$, with W_p a minimal cover of Q_{p-2}. The verification goes as above.

Sufficiency. We proceed by induction on m. The result is trivially true for $m=1$. Assume the result is known for m, and let W satisfy (i)–(iv) for $m+1$, i.e., for $p=2m+2<2\lfloor n/2 \rfloor$. Since G is a forced p-win, at least one non-winning $(p-2)$-set is covered by W_p (Lemma 2.1). If more than one $(p-2)$-set were covered, then Theorem 2.1 would imply $|W_p| \geqslant n-2m-1$, which together with $p<2\lfloor n/2 \rfloor$ would violate (iii). Thus W_p covers exactly one $(p-2)$-set, which is non-winning; call it Q_{p-2}. Clearly, if play reaches some $S_{p-2} \neq Q_{p-2}$, then either S_{p-2} is a winning set or P1 can draw the game. Also, G satisfies condition (a) for $2i=p-2$.

Now consider $W'=(W-W_p) \cup \{Q_{p-2}\}$. By construction it satisfies (ii), (iii), (iv). To show it also satisfies (i), observe that any play of $G'=G(V, W')$ can be interpreted as a partial play of the forced p-win $G(V, W)$, in which Q_{p-2} is the only $(p-2)$-set covered by W_p. Thus P2 can force the play either to terminate on some level $\lambda<p-2$ with a member of $W_\lambda=W'_\lambda$, or else to reach level $p-2$ with some member of $W_{p-2} \cup \{Q_{p-2}\}=W'_{p-2}$, and (since G is a forced p-win) the latter scenario is in fact possible. These observations prove that G' is a forced $(p-2)$-win, i.e., W' satisfies (i).

Thus the induction hypothesis applies to G': there is a chain of subsets $\emptyset=Q_0 \subset Q_2 \subset \cdots \subset Q_{p-4}$, exhibiting W' as $(p-2)$-directive, and with the property (relative to G') stated at the end of the Theorem.

Note that since W' is $(p-2)$-directive with the sets Q_{2i} as above, $G(V, W', Q_{p-4})$ must be a forced 2-win and so $W'_{p-2}=W_{p-2} \cup \{Q_{p-2}\}$ must cover Q_{p-4}. But from (iv) for $m+1$ and from Lemma 2.2, W_{p-2} is too small to cover any $(p-4)$-set. The last two sentences imply $Q_{p-4} \subset Q_{p-2}$, so that Q_{p-2} can be used to extend the chain of subsets.

G satisfies condition (b) for each $2i < p-4$ because G' does (and $W_{2i+2} = W'_{2i+2}$), and does so for $2i = p-4$ because, by (a) for G', $G(V, W', Q_{p-4})$ is a forced 2-win. To see that G satisfies condition (a) for each $2i < p-2$, first observe that each play of $G(V, W, Q_{2i})$ begins with a play of the forced $(p-2-2i)$-win $G(V, W', Q_{2i})$, which P2 can therefore cause to terminate either in some member of $W' - \{Q_{p-2}\} = W - W_p$, or else in Q_{p-2} which is covered by W_p. In either case, a win in $\leqslant p-2i$ moves for P2 in $G(V, W, Q_{2i})$ results. To show that P1 can cause the second case to occur (implying $G(V, W, Q_{2i})$ is a forced $(p-2i)$-win), note that by the induction hypothesis applied to the Theorem's last statement, P1 by conforming to the sets Q_{2j} can 'force' a trajectory in G' that passes through Q_{2i} to reach Q_{p-4}, and then can choose either member of $Q_{p-2} - Q_{p-4}$.

We now have shown that W is p-directive. By the arguments of the first paragraph in the proof, $|W|$ is minimal, so W is a p-filter.

Finally, suppose that in a play of G, $S_{2i} \neq Q_{2i}$ for some i. Then if $S_{2i} \in W_{2i}$ we are done, so suppose not. If $2i = p-2$, then since Q_{p-2} is the only $(p-2)$-set covered by W_p, P1 can draw $G(V, W, S_{2i})$. If $2i < p-2$, then by the induction hypothesis P1 can draw $G(V, W', S_{2i})$ and so in particular can cause play from S_{2i} to reach some $S_{p-2} \neq Q_{p-2} \in W'_{p-2}$; since S_{p-2} is not covered by W_p, P1 can draw $G(V, W, S_{2i})$. Thus the induction has been extended to the Theorem's last statement. □

Remark. In the sufficiency argument of the above theorem it was necessary to exclude the case $n = 2m+1$, by requiring $p < 2\lfloor n/2 \rfloor$. There are generic counterexamples to the sufficiency assertion for that case.

Note that the preceding 'necessity' proof also established the *existence* of p-filters for all n and all even $p = 2m < n$. By the Theorem's conditions (iii) and (iv), all p-filters W for $G(n, W)$ have the same cardinality

$$f(n, m) = \sum_{i=1}^{m} (k-i) + 1 = mk - (m+2)(m-1)/2. \qquad (\dagger)$$

We now formalize the previously announced conjecture, that unless $\Delta = n - 2m$ is small, the p-filters are optimal for problem $\Pi(n, p)$. A stronger version (see (c) below) would require that p-filters be the *only* optimal solutions.

Filter Conjecture. Assume $G(n, W)$ is a forced $2m$-win.

(a) If $\Delta = 3$, then for $m \geqslant 3$ we have $|W| \geqslant f(n, m) - 1$, with equality possible. For $m \leqslant 2$ we have $|W| \geqslant f(n, m)$ with equality possible.

(b) If $\Delta \geqslant 4$, then $|W| \geqslant f(n, m)$ with equality possible.

(c) If $\Delta \geqslant 5$, then equality holds in (b) iff W is a $2m$-filter.

Note that the 'equality possible' statement in (b), and the second one in (a), follow from the existence of p-filters. To verify the first one in (a) (where $n = 2m+3$), we take the Q_{2i}'s as in the preceding 'necessity' proof, but now choose

$$W_{2i+2} = \{Q_{2i} \cup \{2j-1, 2j\} : j = i+2, \ldots, k-1\} \\ \cup \{Q_{2i} \cup \{2k-2, 2k-1\}\} \quad \text{for } i \leq m-4,$$

consistent with (iv) in Theorem 2.3, but with the winning sets on levels $2m-4$, $2m-2$, $2m$ forming a forced 6-win using $9 = f(9, 3) - 1$ winning sets in the continuation game $G(V, W, Q_{p-6})$. Such a forced 6-win is exhibited later, at the start of the proof of Theorem 3.3.

The following 'generic example' shows that (b) cannot be extended to $\Delta = 4$ for $m \geq 2$. Since n must be even, the vertices are paired off:

$$V = \{1, 2\} \cup \{3, 4\} \cup \cdots \cup \{n-1, n\}.$$

Player 2 plays the pairing strategy with respect to these pairings, responding to a choice of one of $\{2i-1, 2i\}$ by a choice of the other. W consists of the entire collection of chosen sets on level $2m$ that are possible under this strategy, i.e., all unions of $m = k-2$ pairs from among the k above. This yields a forced $2m$-win in which, although W is not (for $m > 1$) a $2m$-filter, $|W|$ agrees with the number of winning sets in a $2m$-filter, which is

$$m \cdot k - (m+2)(m-1)/2 = (k-2)k - k(k-3)/2 = k(k-1)/2.$$

Similarly, that (a) cannot be extended to $\Delta = 2$ (for $m \geq 2$) is shown by a simple modification of the preceding example: take W to consist of all unions of $m = k-1$ pairs from among the k listed. This yields a forced $2m$-win with

$$|W| = k < (k-1)k - (k+1)(k-2)/2 = f(2k, 2k-2)$$

for $m = k-1 \geq 2$. We next show that this construction characterizes the optimal solutions to $\Pi(2m+2, 2m)$, thus disposing of the case $\Delta = 2$ just below the range ($\Delta \geq 3$) of the Filter Conjecture.

Theorem 2.4. *If $G(n, W)$ is a forced $2m$-win, $n = 2m+2$, then $|W| \geq k$ with equality only in the (attainable) case $|W| = |W_{2m}| = k$.*

Proof. By Theorem 2.2 each element must appear in at least $\lfloor p/2 \rfloor = k-1$ winning sets, so that (using indicator-function notation)

$$I = \sum_{v \in V} \sum_{w \in W} I(v \in w) \geq n(k-1) = k(2k-2).$$

But since (Lemma 2.3) we may assume all winning sets lie on levels $\leq 2m = 2k-2$, we also have

$$I = \sum_{w \in W} \sum_{v \in V} I(v \in w) \leq |W|\,(2k-2).$$

with equality iff $W = W_{2m}$. Chaining these inequalities yields the result, with the previous construction showing that equality can indeed be attained. □

We conclude this section by disposing of the remaining situation, $\Delta = 1$, below the range of the Filter Conjecture:

Theorem 2.5. *If $G(n, W)$ is a forced $2m$-win, $n = 2m + 1$, then $|W| \geqslant k$ with equality only in the (attainable) case $|W| = |W_{2m}| = k$.*

Proof. Since $k = m + 1$ as in the last proof, the preceding argument again yields $|W| \geqslant k$ with equality iff $|W| = |W_{2m}| = k$. To show that this last situation can arise, take $V = \{1, 2, \ldots, 2k - 1\}$ and let W consist of the k sets $V - \{2i - 1\}$, $i = 1, 2, \ldots, k$. P2 can force a win by choosing an *even* element so long as any remain unchosen. After move $n - 1 = 2m$, S_{2m} must coincide with some member of W. □

3. Validation of low order cases

We now proceed to the verification of the Filter Conjecture for small m, beginning in the next theorem with $m = 1$. Our arguments are somewhat eclectic (much less so than their earliest versions!), but get the job done. As before, we set $k = \lceil n/2 \rceil$ and $p = 2m < n$. A *standing hypothesis* throughout the following proofs is that $G = G(n, W)$ is a forced $2m$-win for which $|W|$ is minimum, i.e., a solution to $\Pi(n, p)$. Since a p-filter is feasible for $\Pi(n, p)$, we know $|W| \leqslant f(n, m)$. We seek to prove that $|W| \geqslant f(n, m)$ and for (b) of the Conjecture, that $|W|$ must in fact be a p-filter. (The first goal is subject to the single exception noted in (a) of the Conjecture.)

Lemma 3.1. *$|W_{2m}| \geqslant k - m + 1$, $W_\lambda = \emptyset$ for odd λ or $\lambda > p$, and no member of any W_λ is covered by $W_{\lambda+2}$.*

Proof. The second and third assertions follow from Lemma 2.3, the first one from Lemmas 2.2 (with $\lambda = p - 2$) and 2.1. □

Theorem 3.1. *For $m = 1$, and all $\Delta \geqslant 1$, $|W| = f(n, 1)$ and W is a 2-filter.*

Proof. By Lemma 3.1, $|W_2| \geqslant k = f(n, 1)$, and all other $W_\lambda = \emptyset$. The result now follows from Theorems 2.3–2.5. □

Theorem 3.2. *For $m = 2$, the Filter Conjecture holds. If also $\Delta = 4$, then either W is a 4-filter or $W = W_4$, with the second case possible.*

Proof. Consider a forced 4-win $G = G(n, W)$, with $n \geqslant 7$ and $|W|$ minimum. By Lemma 3.1, W has winning sets only on levels 2 and 4, and $|W_4| \geqslant k - 1$. Since $|W| = |W_2| + |W_4| \leqslant f(n, 2) = 2k - 2$, we have $|W_2| \leqslant k - 1$; if $|W_2| = k - 1$, then W_2 and W_4 have the correct sizes for Theorem 2.3 to assure a 4-filter.

Therefore, assume $|W_2| \leqslant k-2$. This implies $|\bigcup W_2| \leqslant n-3$, so that there are more than two elements which P1 can choose at move 1 without losing next move. Thus there are at least two 2-sets that must be covered by W_4. This will be shown to yield a contradiction unless $n=7$ and $|W|=6$, or $n=8$ and $|W|=|W_4|=6$, precisely the situations (for $\Delta=3, 4$) permitted by the theorem. That the second of these situations can occur is shown by the 'generic example' following the Filter Conjecture.

By Theorem 2.1 the assumption requires $|W_4| \geqslant 2k-4$ ($2k-3$, for even n). Since $|W| \leqslant 2k-2$, it follows that $|W_2| \leqslant 2$ (with strict inequality for even n). Our argument, using indicator-function notation, will proceed by deriving a lower bound for the quantity

$$4\,|W_4| = \sum_{w \in W_4} \sum_{v \in V} I(v \in w) = \sum_{v \in V} \sum_{w \in W_4} I(v \in w).$$

For each element $x \in \bigcup W_2$ and $v \in V - \bigcup W_2$, P1 can (by choosing v as first move) assure that S_3 contains $\{v, x\}$, and so x must lie in a winning 4-set with each of the $n - |\bigcup W_2|$ members v of $V - \bigcup W_2$, and thus in a least $\lceil (n - |\bigcup W_2|)/3 \rceil$ members of W_4. Each element $y \in V - \bigcup W_2$, as a possible first move by P1, must lie in at least one 2-set that is covered at level 4, and therefore (Lemma 2.2) must lie in at least $k-1$ winning 4-sets. Combining these observations yields

$$4\,|W_4| \geqslant |\bigcup W_2|\,\lceil (n - |\bigcup W_2|)/3 \rceil + (n - |\bigcup W_2|)(k-1).$$

If $|W_2| = 0$, this bound implies

$$|W_4| \geqslant \lceil n(k-1)/4 \rceil \geqslant 2k-2 \quad (\text{if } n \geqslant 7),$$

with equality possible only for $n = 7, 8$. Strict inequality yields the desired contradiction since $|W| \leqslant 2k-2$, while equality yields one of the stipulated exceptions.

Next suppose $|W_2| = 1$, so that $|W_4| \leqslant 2k-3$. Since $n \geqslant 6$, we have $\lceil (n - |\bigcup W_2|)/3 \rceil \geqslant 2$, so that the bound yields

$$|W_4| \geqslant \left\lceil \frac{4 + (n-2)(k-1)}{4} \right\rceil = 1 + \lceil (n-2)(k-1)/4 \rceil.$$

For $n > 7$ this yields $|W_4| \geqslant 2k-2$, the desired contradiction. And for $n = 7$ it yields $|W_4| \geqslant 5$ so that $|W| \geqslant 6$, as desired.

Finally, suppose $|W_2| = 2$, so that $|W_4| \leqslant 2k-4$ and $3 \leqslant |\bigcup W_2| \leqslant 4$. If $n > 7$, or $n = 7$ and $|\bigcup W_2| = 3$, then $\lceil (n - |\bigcup W_2|)/3 \rceil \geqslant 2$, so that the bound gives

$$\begin{aligned} 4\,|W_4| &\geqslant 2\,|\bigcup W_2| + (n - |\bigcup W_2|)(k-1) \\ &= n(k-1) - (k-3)|\bigcup W_2| \geqslant (n-4)(k-1) + 8, \end{aligned}$$

implying

$$|W_4| \geqslant \lceil (n-4)(k-1)/4 \rceil + 2.$$

For $n \geq 7$ this yields $|W_4| \geq 2k-3$ and thus the desired contradiction. For $n=7$ and $|\bigcup W_2| = 4$ the bound yields $|W_4| \geq 4$, so that $|W| \geq 6$ as desired. Hence the theorem is proved. □

Remark. The last Theorem's second assertion does not extend to $\Delta = 3$. Appendix I of [12] exhibits multiple configurations of forced 4-wins for $n=7$, with $|W| = 6$, which exhibit the possibilities $|W_2| = 0, 1, 2, 3$.

From now on, we may assume $m > 2$. It proves economical to introduce here the following definitions and Lemma, which refer to the responses to which P2 is limited if P1 chooses some $v \in V - \bigcup W_2$ as first move.

Definition. For $v \in V - \bigcup W_2$, let $R_v = \{y \in V - \{v\} : G(V, W, \{v, y\})$ a forced win$\}$. Also set $R = \bigcup \{\{v\} \cup R_v : v \in V - \bigcup W_2\}$, and let U denote the set of elements in R^c which do not lie in any 4-set Q_4 such that $G(V, W, Q_4)$ is a forced win. (Thus $V - \bigcup W_2 \subseteq R \subseteq U^c$.)

Lemma 3.2. *Each $u \in U$ lies in at least $\lceil |R|/3 \rceil$ members of W_4, each $x \in R$ in at least $\lceil |U|/2 \rceil$ members. If $|W_2| \leq k-2$, then $|R| \geq 3$ (4, for even n), and $|W_4| \geq |U|$.*

Proof. Consider any $u \in U$. For any $x \in V - \bigcup W_2 \subseteq U^c$, since G is a forced p-win, R_x is nonempty, so that $G(V, W, \{x, y\})$ is a forced win (for P2, by Lemma 3.1) for each $y \in R_x$. The definition of U requires that $\{x, y, u\}$ lie in some winning 4-set. So u lies in some member of W_4 together with each $x \in V - \bigcup W_2$ and some other element of R. Next consider any $x \in R - (V - \bigcup W_2)$; then $x \in R_v$ for some $v \in V - \bigcup W_2$ and it follows as above that $\{v, x, u\}$ lies in some winning 4-set; i.e., each u lies in some member of W_4 together with each $x \in R - (V - \bigcup W_2)$ and some other element of R. Combining these results shows that for each $u \in U$ and $x \in R$, $\{x, u\}$ lie together in a winning 4-set containing at least one element of $R - \{x\}$. This yields the Lemma's first assertions, which in turn imply:

$$4|W_4| = \sum_{w \in W_4} \sum_{v \in V} I(v \in w) = \sum_{v \in V} \sum_{w \in W_4} I(v \in w) \geq |R| \cdot \lceil |U|/2 \rceil + |U| \cdot \lceil |R|/3 \rceil.$$

Now assume $|W_2| \leq k-2$. Since $V - \bigcup W_2 \subseteq R$, we have

$$|R| \geq n - |\bigcup W_2| \geq n - 2|W_2| \geq (2k-1) - 2(k-2) = 3, \text{ (4, for even } n)$$

as desired. If $|R| \geq 4$, then the preceding lower bound on $4|W_4|$ is $\geq 4\lceil |U|/2 \rceil + 2|U| \geq 4|U|$, so that $|W_4| \geq |U|$. And if $|R| = 3$, then the above string of inequalities yielding $|R| = 3$ must be 'tight', so that $V - \bigcup W_2 = R = \{r(1), r(2), r(3)\}$. Without loss of generality assume $r(2) \in R_{r(1)}$. The set $R_{r(3)}$ must contain at least one of $\{r(1), r(2)\}$; assume the former. Then the last paragraph's results show that for each $u \in U$, $\{r(1), r(2), u\}$ and $\{r(3), r(1), u\}$ each lie in a winning 4-set. If d denotes the number of members of W_4 that contain all of R, it follows that

$$|W_4| \geq d + 2\lceil(|U| - d)/2\rceil \geq |U|,$$

completing the proof. $\square$

With the inductive foundation laid, we can add a further standing (induction) hypothesis: *the Filter Conjecture holds for* $m - 1$.

Lemma 3.3. *Suppose* W_{2m} *covers exactly one* $(2m-2)$*-set* Q_{2m-2}. *If* $\Delta \geq 3$, *or if* $m = 2$, *then* W *is a* $2m$*-filter.*

Proof. By minimality of W, Q_{2m-2} is not in W. Let $W' = (W - W_{2m}) \cup \{Q_{2m-2}\}$. Because G is a forced $2m$-win, P2 can force $S_2 \in W_2$ or $S_4 \in W_4$ or $\cdots$ or $S_{2m-2} \in W_{2m-2}$ or $S_{2m-2} = Q_{2m-2}$, and P1 can force the last of these to arise. It follows that $G' = G(n, W')$ is a forced $(2m-2)$-win.

If $\Delta \geq 3$, then $n - (2m-2) \geq 5$ and so (b) and (c) of the Filter Conjecture apply to G'; by Theorem 3.1 they also apply if $m = 2$ (even if $\Delta = 1, 2$). By (b) and Lemma 3.1, we have

$$f(n, m-1) \leq |W'| = |W| + 1 - |W_{2m}| \leq |W| - (k - m)$$

or equivalently $|W| \geq f(n, m)$. Since equality holds, we must have $|W_{2m}| = k - m + 1$ and $|W'| = f(n, m-1)$, the latter implying by (c) that W' is a $(2m-2)$-filter. Thus $|W_{2i}| = |W'_{2i}| = k - i$ for $i < m - 1$, while $|W_{2m-2}| = |W'_{2m-2}| - 1 = (k - (m-1) + 1) - 1$. That W is a $2m$-filter now follows from Theorem 2.3, except when $m = 2$ and $n = 5$. In that case W_4, in covering Q_2, must include at least two 4-sets each containing Q_2, and thus must cover some 2-set other than Q_2 (contradicting the Lemma's hypothesis). $\square$

Theorem 3.3. *For* $m = 3$, *the Filter Conjecture holds. If also* $\Delta = 3$, *then* $|W_2| = 3$ *and* $|W_4| = 0$.

Proof. We first verify the assertion in (a) of the Filter Conjecture, that forced 6-wins with $n = 9$ and $|W| = 9$ exist. In the following example, which conforms to the theorem's second statement, we indicate P2's winning strategy by listing the possibilities for S_2 and S_4 in a 'best-play' realization:

$$W_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, \quad S_2 = \{\{7, 8\}, \{8, 9\}\}$$

$$S_4 = \{\{1, 7, 8, 9\}, \{2, 7, 8, 9\}, \{3, 7, 8, 9\}, \{4, 7, 8, 9\}, \{5, 7, 8, 9\}, \{6, 7, 8, 9\}\}$$

$$W_6 = \{\{1, 2, 3, 7, 8, 9\}, \{1, 2, 4, 7, 8, 9\}, \{1, 5, 6, 7, 8, 9\}, \{2, 5, 6, 7, 8, 9\}, \{3, 4, 5, 7, 8, 9\}, \{3, 4, 6, 7, 8, 9\}\}.$$

Now consider a forced 6-win $G = G(n, W)$, with $n \geqslant 9$, for which $|W|$ is minimum. We are to prove that $|W| \geqslant f(n, 3) = 3k - 5$ (with the stipulated exception for $n = 9$), and that if $n \geqslant 11$ then equality holds only if W is a 6-filter.

By Lemma 3.1, W has winning sets only on levels 2, 4 and 6, and $|W_6| \geqslant k - 2$. We have $k \geqslant 5$ and $|W_2 \cup W_4 \cup W_6| \leqslant 3k - 5$ ($\leqslant 9$ if $n = 9$). By Lemma 2.1, at least one 4-set is covered by W_6, and by Lemma 3.3 we can assume that at least two 4-sets are covered by W_6. By Theorem 2.1, it follows that $|W_6| \geqslant 2k - 6$ ($2k - 5$ if n is even), implying that $|W_2 \cup W_4| \leqslant k + 1$ (k, if n is even; 5, if $n = 9$).

Claim. *No 2-set is covered by* W_4.

Proof of Claim. If more than one 2-set is covered by W_4, Theorem 2.1 yields $|W_4| \geqslant 2k - 4$ ($2k - 3$, for even n). The sentence before the claim yields $|W_4| \leqslant k + 1$ (k, for even n; 5, if $n = 9$). For even n, the resulting inequality $2k - 3 \leqslant k$ contradicts the fact that $k \geqslant 5$. For odd n, the resulting inequality $2k - 4 \leqslant k + 1$ requires $k = 5$ and $|W_4| = 6$, but then $n = 9$ and $|W_4| \leqslant 5$ is contradicted.

Now suppose W_4 covers a *unique* 2-set Q_2. The coverage implies by Lemma 2.2 that $|W_4| \geqslant k - 1$, so that $|W_2| \leqslant 2$ ($|W_2| \leqslant 1$, if n is even or $n = 9$). It follows that

$$
\begin{aligned}
I_{46} &= \sum_{w \in W_4 \cup W_6} \sum_{v \in V} I(v \in w) = 4\,|W_4| + 6\,|W_6| \\
&\leqslant 4 \cdot |W_4| + 6(3k - 5 - |W_2| - |W_4|) \\
&= 6(3k - 5) - 2\,|W_4| - 6\,|W_2| \leqslant 6(3k - 5) - 2(k - 1) - 6\,|W_2|,
\end{aligned}
$$

which for $n = 9$ sharpens to $I_{46} \leqslant 46 - 6\,|W_2|$.

The argument proceeds by deriving a *lower* bound for I_{46}. Each of the elements in $(\bigcup W_2) - Q_2$ must, by Theorem 2.2, lie in at least one member of W_6. Each of the two elements in Q_2 must by Lemma 2.2 lie in at least $k - 1$ members of W_4, and must by Theorem 2.2 lie in at least one member of W_6. Now consider the $n - 2 - |(\bigcup W_2) - Q_2|$ elements in $V - Q_2 - \bigcup W_2$. Since Q_2 is the only 2-set covered by W_4 and G is a forced 6-win, for each $v \in V - Q_2 - \bigcup W_2$ the set R_v defined before Lemma 3.2 has at least one member y. The continuation game $G(V, W, \{v, y\})$ is a forced 4-win on the $n - 2$ elements of $V - \{v, y\}$, and so by Theorem 3.2 has at least $2(k - 1) - 2 = 2k - 4$ winning sets. Thus v lies in at least $2k - 4$ members of $W_4 \cup W_6$. Combining these observations gives

$$
\begin{aligned}
I_{46} &= \sum_{v \in V} \sum_{w \in W_4 \cup W_6} I(v \in w) \geqslant |(\textstyle\bigcup W_2) - Q_2| \\
&\quad + 2k + (n - 2 - |(\textstyle\bigcup W_2) - Q_2|)(2k - 4) \\
&\geqslant (n - 2)(2k - 4) + 2k - (2k - 5)\,|\textstyle\bigcup W_2|.
\end{aligned}
$$

For $n = 9$, chaining the upper and lower bounding inequalities on I_{46} yields

$$5\,|\textstyle\bigcup W_2| - 6\,|W_2| \geqslant 6,$$

which is false since $|W_2| \leqslant 1$. For $n \geqslant 10$, chaining gives an inequality rewritable as

$$(2k-5)|\bigcup W_2| - 6\,|W_2| \geqslant 2(n-9)(k-2).$$

This is false when $|W_2| = 0$ (since $n \geqslant 10$). For $|W_2| = 1$, when $n \geqslant 11$ it yields

$$4k - 16 \geqslant 2((2k-1)-9)(k-2)$$

which is false, and for $n = 10$ it also yields a false result. For $|W_2| = 2$ (so that $n = 2k-1 \geqslant 11$), since $|\bigcup W_2| \leqslant 4$ the inequality gives

$$(2k-5)4 - 6 \cdot 2 \geqslant 2((2k-1)-9)(k-2),$$

yielding a contradiction except for equality holding when $n = 11$ and $|\bigcup W_2| = 4$. But in this remaining case the lower bound can be sharpened (so that a contradiction again results): each element of $\bigcup W_2 - Q_2$ lies in only one member of W_2 yet (Theorem 2.2) in at least 3 members of W, hence in at least 2 members of $W_4 \cup W_6$. □ (Claim)

Consequences of the Claim. We may now assume that no 2-set Q_2 is covered by W_4. Because G is not a forced 2-win, there exist elements $v \in V - \bigcup W_2$, each with at least one 'response' $y \in R_v$ such that $G(V, W, \{v, y\}) = G(V - \{v, y\}, W_v)$ is a forced 4-win. As in the proof of the claim, Theorem 3.2 yields $|W_4 \cup W_6| \geqslant |W_v| \geqslant 2k-4$, which since $|W| \leqslant 3k-5$ (9, if $n = 9$), implies $|W_2| \leqslant k-1$ (3, if $n = 9$). We will first treat the case $|W_2| = k-1$, then the cases $|W_2| \leqslant k-2$.

If $|W_2| = k-1$ then the above gives $|W_4 \cup W_6| = 2k-4$, and thus $|W| = 3k-5$ as desired. It also gives $|W_v| \leqslant 2k-4$. If further $n \geqslant 11$, then Theorem 3.2 implies that W_v is a 4-filter, so that $|W_4| \geqslant |(W_v)_2| = k-2$ and $|W_6| \geqslant |(W_v)_4| = k-2$. Thus equality holds throughout, so that (if $n \geqslant 11$) Theorem 2.3 guarantees W is a 6-filter as desired.

In treating the remaining cases $|W_2| \leqslant k-2$, we first use Lemma 3.2 to lower bound the previously-defined quantity I_{46}. Each element in R, whether a $v \in V - \bigcup W_2$ or a $y \in R_v$ for some $v \in V - \bigcup W_2$, lies in some $S_2 = \{v, y\}$ for which the continuation of G from S_2 is a forced 4-win, and therefore (cf. the proof of the claim) must lie in at least $2k-4$ members of $W_4 \cup W_6$. Each element in $R^c - U$ lies (by the definition of U) in some 4-set that is covered by W_6, and so by Lemma 2.2 must lie in at least $k-2$ members of W_6. Each element in U lies by Theorem 2.2 in at least one member of W_6, and by Lemma 3.2 in at least $\lceil |R|/3 \rceil \geqslant 1$ members of W_4. Combining these observations gives

$$I_{46} \geqslant |R|\,(2k-4) + (n - |R| - |U|)(k-2) + 2\,|U| = (n + |R|)(k-2) - (k-4)\,|U|,$$

which since $V - \bigcup W_2 \subseteq R$, $|\bigcup W_2| \leqslant 2\,|W_2|$ and (Lemma 3.2) $|U| \leqslant |W_4|$, gives

$$I_{46} \geqslant (2n - 2\,|W_2|)(k-2) - (k-4)\,|W_4|.$$

On the other hand, since $|W| \leqslant 3k-5$ (9, for $n = 9$), we have as in the claim's proof

$$I_{46} \leqslant 6(3k-5) - 6\,|W_2| - 2\,|W_4|,$$

with $I_{46} \leqslant 54 - 6\,|W_2| - 2\,|W_4|$ if $n = 9$. Chaining the lower and upper-bound inequalities on I_{46} for $n = 10$ gives a result rewritable as

$$(2k-10)\,|W_2| + (k-6)\,|W_4| \geqslant 2n(k-2) - 6(3k-5). \qquad (\ddagger)$$

For $n \geqslant 11$ (hence $k \geqslant 6$), the relations $|W_2 \cup W_4| \leqslant k+1$ and $n \geqslant 2k-1$ now yield from (‡)

$$(2k-10)\,|W_2| + (k-6)(k+1-|W_2|) \geqslant (4k-2)(k-2) - 6(3k-5),$$

which gives a contradiction since $|W_2| \leqslant k-2$. If $n = 10$ (hence $k = 5$), then (‡) reads: $-|W_4| \geqslant 0$, implying $|W_4| = 0$. Here all inequalities leading to (‡) must also be 'tight', so that in particular $|W| = 3k-5 = 10$ as desired.

The only remaining case is $n = 9$ ($k = 5$), when the chaining yields $-|W_4| \geqslant 0$. Hence again $|W_4| = 0$ and all inequalities involved in the bounding must be 'tight'. In particular, $|W| = 9$ (as desired), a previous expression for I_{46} becomes $6(9 - |W_2|)$, and we have $|\bigcup W_2| = 2\,|W_2|$, $R = V - \bigcup W_2$ and $R = 9 - 2\,|W_2|$. We are in the case $|W_2| \leqslant k-2 = 3$, and need only prove equality holds; this will be done by contradiction.

Suppose then that $|W_2| \leqslant 2$, implying $|R| \geqslant 5$. Consider any $v \in R^c$. P1 can first choose any $z \in R$ to assure that $\{z, v\} \subseteq S_3$. Since $|W_4| = 0$, P2 can make play from S_3 yield a 6-move win, and so $\{z, v\}$ lies in a 4-set covered by W_6. For fixed v, z can be chosen in $\geqslant 5$ ways, so v lies in at least 2 4-sets covered by W_6. By Theorem 2.1, v lies in at least $n - 5 = 4$ members of W_6.

Together with the consequence $|U| = 0$ of Lemma 3.2, this sharpens the previous lower-bounding argument for I_{46} to

$$\begin{aligned} I_{46} &\geqslant 6\,|R| + 4\,|R^c| = 4n + 2\,|R| \\ &= 4n + 2(n - 2\,|W_2|) = 54 - 4\,|W_2|. \end{aligned}$$

Comparison with $I_{46} = 6(9 - |W_2|)$ yields $|W_2| = 0$. Thus $W = W_6$. Since the lower bound now coincides with $6\,|W_6| = 6\,|W| = 54$, it follows that every element must lie in *exactly* 6 members of W_6.

For each $v \in V - \bigcup W_2$ and $y \in R_v \subseteq V - \bigcup W_2$, continuation of G from $S_2 = \{v, y\}$ is a forced 4-win on 7 elements, with no winning 2-sets since $|W_4| = 0$, so by Theorem 3.2, S_2 lies in at least 6 members of W_6. Thus v and y must lie in the *same* members of W_6. Since $n - |\bigcup W_2| = 9 - 2\,|W_2|$ is odd, some $v \in V$ must lie in at least 2 such pairs S_2, say $\{v, y\}$ and $\{v, x\}$. Then $X = \{v, x, y\}$ must lie in some 6 members of the 9 members of W_6, and be disjoint from the remaining $|W| - 6 = 3$ members. But the 6-set $V - X$ cannot contain 3 distinct 6-sets. □ (Theorem 3.3)

Remark. Apropos the situations $\Delta = 3, 4$ in the context of the last Theorem: Appendix II of [12] exhibits multiple configurations of forced 6-wins for $n = 9, 10$ with $|W| = 10$.

4. The weak filter theorem

We now give a weakened form of the Filter Conjecture which we are able to prove for *all* cases, not just those of low order. The proof of this Theorem 4.1 is inductive; its 'induction step' will be given immediately after the statement of the Theorem, but the somewhat laborious reasoning needed to establish the 'induction base' is deferred to an appendix, where it appears as Lemmas A and B. This unorthodox sequencing of material was chosen to give preference to intrinsic interest (we hope) over strict logical progression.

Note that in the Set Coincidence game, choosing any element in the complement of some $w \in W$ rules out the possibility that continuation of play will yield $S = w$, and thus in more vivid language *kills* w. We therefore define

$$K = \{\sigma \colon \sigma = w^c, w \in W\}$$

and refer to its members as *kill sets*. As before, $k = \lceil n/2 \rceil$.

Theorem 4.1 (Weak Filter Theorem). *Assume $G(n, W)$ is a forced $2m$-win, with $m \geqslant 4$. If $n \geqslant 2m + 3$, then $|W| \geqslant n + 3$.*

Proof. The proof is by induction on $m \geqslant 4$. As stated above, verification of the 'base case' $m = 4$ is deferred to an appendix (Lemma A). Verification for the special cases $(m, n) = (5, 13)$, $(5, 14)$ is given in the Appendix's Lemma B.

For the induction step, assume the result is true for forced $2q$-wins for all $q \in [4, m]$. Suppose $G = G(n, W)$ is a forced $(2m + 2)$-win with $n \geqslant 2m + 5 \geqslant 13$ and $|W|$ minimum. We wish to prove $|W| \geqslant n + 3$; for a contradiction, assume $|W| \leqslant n + 2$. By Lemma 2.1, W has sets on at most levels $2, 4, \ldots, 2m + 2$.

Claim 1. $|W| - |W_2| \geqslant n + 1$, *so that* $|W_2| \leqslant 1$.

Proof of Claim 1. Some first move v by P1 can prolong play to $2m + 2$ moves. If $u \in R_v$, then $G(V, W, \{v, u\})$ is a forced $2m$-win on $n - 2 \geqslant 2m + 3$ elements. By the induction hypothesis this continuation has at least $(n - 2) + 3 = n + 1$ winning sets, which adjoined to $\{v, u\}$ yield distinct winning sets of G at levels >2. This proves the first assertion, from which the second follows since $|W| \leqslant n + 2$. □ (Claim 1)

Claim 2. *At least* 3 *elements are in* 2 *or more kill sets.*

Proof of Claim 2. All winning sets are of size $\leqslant 2m + 2$, so all kill sets have size at least $n - (2m + 2) \geqslant 5 - 2 = 3$. By Claim 1 there are at least $n + 1$ such sets, so

$$J = \sum_{\sigma \in K} \sum_{v \in V} I(v \in \sigma) \geqslant 3(n + 1).$$

By Theorem 2.2 with $p = 2m + 2 \geqslant 10$, each element lies in at least 5 winning sets and thus in at most $|W| - 5 \leqslant n - 3$ of the $|W|$ kill sets. If at most 2 elements lay in 2 or more kill sets, then at least $n - 2$ would each lie in at most 1 kill set, yielding

$$J = \sum_{v \in V} \sum_{\sigma \in K} I(v \in \sigma) \leqslant 2(n-3) + (n-2) = 3n - 8;$$

this contradicts the preceding lower bound for J. $\square$ (Claim 2)

Consequence of Claims 1 and 2. From Claim 2, and Claim 1's implication $|\bigcup W_2| \leqslant 2$, there must be some $v \in V - \bigcup W_2$ whose initial choice by P1 kills at least 2 winning sets. For a win-preserving response y by P2, the continuation $G' = G(n-2, W')$ of play from $S_2 = \{v, y\}$ must for some $q \leqslant m$ be a forced $2q$-win. Since $|W| \leqslant n + 2$, the choice of v implies that $|W'| \leqslant n$, and since $n - 2 \geqslant 2m + 3 \geqslant 2q + 3$, applying the induction hypothesis to G' shows that $q \leqslant 3$. We now rule out in turn each of the possibililties $q = 1, 2, 3$. Note that $n \geqslant 13$, so that $k \geqslant 7$.

Case: $q = 1$. Then S_2 must be covered by W_4, implying by Lemma 2.2 that $|W_4| \geqslant k - 1$. Next, play of G enforcing a $(2m+2)$-move win yields an S_4 such that the continuation game G_4 of G from S_4 is a forced $(2m-2)$-win on $n - 4$ elements. Note that $n - 4 \geqslant (2m - 2) + 3$ and that the winning sets of G_4 correspond (via adjunction of S_4) to winning sets of G at levels >4. Thus if $m = 4$ then Theorem 3.3 applies to G_4 when $n > 13$ to give $|W - W_2 - W_4| \geqslant 3(k-2) - 5$, yielding

$$|W - W_2| \geqslant 3(k-2) - 5 + (k-1) \geqslant n + 3 \quad (n > 14)$$

which contradicts $|W| \leqslant n + 2$; the cases $(m, n) = (4, 13), (4, 14)$ corresponding to forced 10-wins G on 13, 14 elements are dealt with in the later Lemma B. If $m > 4$, then the induction hypothesis applies to G_4 to yield $|W - W_2 - W_4| \geqslant (n-4) + 3$, so that

$$|W - W_2| \geqslant (n-4) + 3 + (k-1) = n + (k-2),$$

again contradicting $|W| \leqslant n + 2$.

Case: $q = 2$. Then application of Theorem 3.2 to G' yields $|W_4 \cup W_6| \geqslant 2(k-1) - 2$. Play of G enforcing a win in $(2m+2)$ moves yields an S_6 such that the continuation game G_6 of G from S_6 is a forced $(2m-4)$-win on $n - 6$ elements, where $(n-6) \geqslant (2m-4) + 3$. Thus if $m = 4$ then Theorem 3.2 applied to G_6 yields $|W - W_2 - W_4 - W_6| \geqslant 2(k-3) - 2$, so that

$$|W - W_2| \geqslant 2(k-3) - 2 + 2(k-1) - 2 = 4k - 12,$$

which contradicts $|W| \leqslant n + 2$ unless $n = 14$, where Lemma B applies to G. If $m = 5$ so that $n \geqslant 15$, then Theorem 3.3 applied to G_6 when $n > 15$ yields $|W - W_2 - W_4 - W_6| \geqslant 3(k-3) - 5$, so that

$$|W - W_2| \geqslant 3(k-3) - 5 + 2(k-1) - 2 \geqslant n + 3 \quad (n > 15)$$

contradicting $|W| \leqslant n+2$; for $n=15$ Theorem 3.3 yields $|W - W_2 - W_4 - W_6| \geqslant 9$ and the contradiction again occurs. And if $m>5$ then the induction hypothesis applies to G_6 to give $|W - W_2 - W_4 - W_6| \geqslant (n-6)+3$, yielding

$$|W - W_2| \geqslant (n-6)+3+2(k-1)-2 = n+(2k-7),$$

again contradicting $|W| \leqslant n+2$.

Case: $q=3$. Applying Lemmas 2.1 and 2.2 to G, with $p=2m+2$ and $\lambda=2m$, yields

$$|W_{2m+2}| \geqslant k-m \geqslant (n/2)-(n-5)/2,$$

so that $|W_{2m+2}| \geqslant 3$. Since $n-2 \geqslant 11$, application of Theorem 3.3 to G' yields $|W_4 \cup W_6 \cup W_8| \geqslant 3(k-1)-5$. Summing these results gives

$$|W - W_2| \geqslant 3+3(k-1)-5 \geqslant n+3 \quad (n \neq 14),$$

yielding the desired contradiction to $|W| \leqslant n+2$ unless $n=14$, which implies $m=4$. In that case the later Lemma B applies to G to yield $|W| \geqslant 17$ as desired. □ (Theorem 4.1)

Appendix

We turn now to the results (Lemmas A and B below), about the Set Coincidence Game, that are needed as basis for the induction proof of Theorem 4.1. The (refereed) proof of Lemma A is available from the authors; involving more elaborate applications of the arguments found in this paper's other proofs, it has been deleted for the sake of brevity.

Lemma A. *If* $G(n, W)$ *is a forced* 8-*win with* $n \geqslant 11$, *then* $|W| \geqslant n+3$.

Lemma B. *A forced* 10-*win on* $n=13, 14$ *elements has* $|W| \geqslant n+3$.

Proof. Let $G=G(n, W)$ be such a forced 10-win, with $|W|$ minimum. Note that $k=7$. Thus by Lemma 3.1, W lies entirely on levels 2, 4, 6, 8, 10. Assume for a contradiction that $|W| \leqslant n+2$.

The continuation of G from some suitable 4-set Q_4 is a forced 6-win G_4 on $n-4 \geqslant 9$ elements, which by Theorem 3.3 implies

$$|W_6 \cup W_8 \cup W_{10}| \geqslant 10 - I(n=13). \qquad (*)$$

Also, as in the proof of Theorem 4.1 with Lemma A replacing 'the induction hypothesis', we have

Claim 1. $|W - W_2| \geqslant n+1$, *so that* $|W_2| \leqslant 1$.

The argument proceeds via upper and lower bounds on the quantity

$$\begin{aligned}
I_{468(10)} &= \sum_{v\in V} \sum_{w\in W-W_2} I(v \in w) = \sum_{j=2}^{5} (2j)\,|W_{2j}| \\
&= 10\Big(|W| - \sum_{j=1}^{4} |W_{2j}|\Big) + \sum_{j=2}^{4} 2j|W_{2j}| \\
&\leqslant 10(n+2) - 10\,|W_2| - 6\,|W_4| - 4\,|W_6| - 2\,|W_8|. \qquad (**)
\end{aligned}$$

Note that each $v \in V$, as first move by P1, leads with quickest-win play by P2 either to (a) a win on move 2, or else an $(n-2)$-element continuation $G(V, W, S_2)$ which can be (b) a forced 2-win, (c) a forced 4-win, (d) a forced 6-win, or (e) a forced 8-win.

Note that (a) applies if and only if v is one of the at most 2 elements in $\bigcup W_2$; by Theorem 2.2 and Claim 1, v lies in at least 4 members of $W - W_2$.

If (b) applies, then by Lemma 2.2 v lies in at least $k-1=6$ members of W_4. Also, if $v \in Q_4$ then Theorem 3.3 applied to G_4 shows that v lies in at least $3(k-2)-5-I(n=13)=10-I(n=13)$ members of $W_6 \cup W_8 \cup W_{10}$, while if $v \in V - Q_4$ then Theorem 2.2 applied to G_4 shows that v lies in at least 3 members of $W_6 \cup W_8 \cup W_{10}$. In either case, v lies in at least 9 members of $W - W_2$.

If (c) applies, then by Theorem 3.2, v lies in at least $2(k-1)-2=10$ winning sets in $W_4 \cup W_6$. Also, continuation of G from a suitable 6-set Q_6 yields a forced 4-win G_6 on $n-4$ elements; if $v \in Q_6$ then Theorem 3.2 applied to G_6 shows that v lies in at least $2(k-2)-2=8$ members of $W_8 \cup W_{10}$, while if $v \in V - Q_6$ then Theorem 2.2 applied to G_6 shows that v lies in at least 2 members of $W_8 \cup W_{10}$. In either case, v lies in at least 12 members of $W - W_2$.

If (d) applies then by Theorem 3.3, v lies in at least $3(k-1)-5=13$ members of $W_4 \cup W_6 \cup W_8$. Also, by Theorem 2.2 v lies in at least 1 member of W_{10}, hence in at least 14 members of $W - W_2$.

Finally, if (e) applies then by Lemma A, v lies in at least $(n-2)+3=n+1$ members of $W - W_2$. Since G is a forced 10-win, this must apply for at least one element v.

Let n_b, n_c, n_d, n_e denote the respective numbers of elements to which each of (b)–(e) apply. Then $n_e \geqslant 1$, and we have

$$I_{468(10)} \geqslant 4\,|\bigcup W_2| + 9n_b + 12n_c + 14n_d + (n+1)n_e. \qquad (***)$$

Claim 2. *No 2-set is covered by W_4.*

Proof of Claim 2. Otherwise, by Lemma 2.2, $|W_4| \geqslant k-1=6$ would hold. Adding this to $(*)$ leads to

$$n+2 \geqslant |W| \geqslant |W - W_2| \geqslant 6 + (10 - I(n=13)) = n+2,$$

so that equality holds throughout. Thus $|W_2| = 0$, so that alternative (a) vanishes, and $|W_4| = 6$. This last relation, plus Theorem 2.1, shows that W_4 covers a *unique* 2-set, Q_2. Alternative (b) above, can hold only for elements in Q_2, so that $n_b \leqslant 2$. Since $|W_2| = 0$, chaining the upper and lower bounds on $I_{468(10)}$ yields (remember $n_e \geqslant 1$)

$$\begin{aligned} 10(n+2) \geqslant I_{468(10)} &\geqslant 9n_b + 12n_c + 14n_d + (n+1)n_e \\ &\geqslant 9n_b + 12(n_c + n_d + n_e) + (n-11)n_e \\ &= 9n_b + 12(n - n_b) + (n-11)n_e \\ &= 12n + (n-11)n_e - 3n_b \geqslant 13n - 17, \end{aligned}$$

contradicting $n \geqslant 13$. □

Consequence of the Claims. By Claim 2, the preceding alternative (b) vanishes, so that by $(***)$, since $n_e \geqslant 1$,

$$\begin{aligned} I_{468(10)} &\geqslant 4\,|\bigcup W_2| + 12(n_c + n_d + n_e) + (n-11)n_e \\ &\geqslant 4\,|\bigcup W_2| + 12(n - |\bigcup W_2|) + (n-11)n_e \geqslant 13n - 11 - 16\,|W_2|. \end{aligned}$$

If some 4-set is covered by W_6, then Lemma 2.2 implies $|W_6| \geqslant 5$, so that $(**)$ yields

$$I_{468(10)} \leqslant 10(n+2) - 10\,|W_2| - 4 \cdot 5.$$

Chaining the most recent upper and lower boundings of $I_{468(10)}$ yields $11 + 6\,|W_2| \geqslant 3n$, a contradiction since $|W_2| \leqslant 1$.

Thus no 4-set is covered by W_6, so that alternative (c) also vanishes. Now $(***)$ plus $n_e \geqslant 1$ gives

$$\begin{aligned} I_{468(10)} &\geqslant 4\,|\bigcup W_2| + 14n_d + (n+1)n_e \\ &= 4\,|\bigcup W_2| + 14(n - |\bigcup W_2|) + (n-13)n_e \geqslant 15n - 13 - 10\,|W_2|, \end{aligned}$$

whereas $(**)$ gives

$$I_{468(10)} \leqslant 10(n+2) - 10\,|W_2|.$$

Chaining the most recent boundings gives $33 \geqslant 5n$, a contradiction. □ (Lemma B)

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